Table 1 Values of λ_0 and β_0 for variousboundary conditions, $\nu=0.3$.

	Clamped-clamped	Clamped-free	Clamped-simply supported	Simply supported simply supported
λ_0	68.8	12.64	53.4	$4\pi^2$
β_0	4.75	1.92	3.90	π

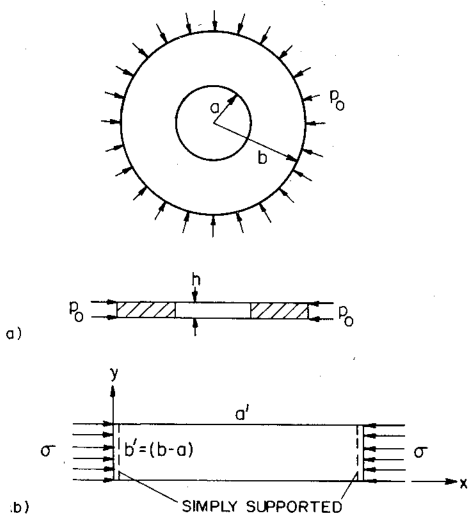

Fig. 1 a) Annular plate under uniform external pressure; b) rectangular strip under uniform compression.

$$X=x/a', \quad Y=y/b', \quad W(X,Y)=w(Y)\sin m\pi X$$

$$(V-T)=\frac{D}{2}\frac{1}{b'^4}\left[\int_0^1\left\{\left(\frac{d^2w}{dY^2}-\left(\frac{m\pi b'}{a'}\right)^2 w\right)^2 + 2(1-\nu)\left(\frac{m\pi b'}{a'}\right)^2\times\left(w\frac{d^2w}{dY^2}+\left(\frac{dw}{dY}\right)^2\right) - \frac{\sigma h b'^2}{D}\left(\frac{m\pi b'}{a'}\right)^2 w^2\right\}dY\right]$$

admissible functions. The curve λ vs β is then plotted from which the value of β at which λ assumes its minimum is obtained. The values of λ and β thus obtained are denoted by λ_0 and β_0 and are given in Table 1.

Note that in deriving the limiting variational problem Eq. (5), the first and the higher-order terms of $(b-a)/a$ in the modified energy expressions are neglected in the case of annulus, whereas no such approximation is involved in the strip problem. Hence the limiting solutions $\sigma h b'^2/D=\lambda_0$ and $m\pi b'/a=\beta_0$ in the strip problem can be interpreted in a useful manner for some finite strips. Whereas no such interpretation is possible in the case of annular plates, since the limiting solutions $(2p_0b^2h/D)(b-a)/(b+a)=\lambda_0$ and $n(b-a)/a=\beta_0$ are exact only in the limit $a\rightarrow b$. Nevertheless, the latter solution is useful in the sense that the value $\beta_0 a/(b-a)$ gives an indication of the number of circumferential waves (n) to be expected in the critical buckling mode of an annular plate.

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Shrouds for Aerogenerators

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Introduction

WIND is a natural energy resource that has heretofore not been widely used because of low energy content. Since the density of atmospheric air is approximately only one-thousandth that of water, a stream of air has only one-thousandth the energy contained in a stream of water of the same flow crosssection and velocity. Furthermore, it can be shown that at maximum efficiency an ideal windmill can use only 59.3% of the energy available in the stream tube that covers the windmill blades while the rest is carried downstream with the wind.[1,2] In real facilities, the efficiency is of course even less, due to aerodynamic and mechanical losses. Past attempts to exploit wind power for generating electricity have used windmills with giant blades in order to obtain practicable amounts of energy.[2-4] For example, in the Smith-Putnam project, the blade diameter was 175 ft.[3] In such an installation, the rotor must turn very slowly, resulting in design problems for the gears necessary to connect the windmill with the generator. Furthermore, in order to ensure a relatively stable power output under "off-design" conditions, it is necessary to use a rotor equipped with a blade pitch control and to keep its axis always parallel to the free wind direction. As a result, in the past, wind power was not economically competitive with other energy sources. To use wind power efficiently, reduce the size of rotors, and to increase the rotational speeds, various combinations of turbines operating inside specially designed shrouds have been investigated in Israel.[5-7] The investigated shrouds (one is shown schematically in Fig. 1) were composed of a bell-shaped intake, a cylindrical section, and a diffuser.

The purpose of the present paper is to present a new approach to shroud design, such that good aerodynamic performance is retained, while the shroud is made more attractive economically. Figure 1 summarizes the results obtained from the first model of the shroud.[6,7] The power ratio r is defined as

$$r=\frac{\frac{1}{2}\rho q_t{}^3 A_t C_D}{0.593\,\frac{1}{2}\rho q_\infty{}^3 A_t}=\frac{27}{16}\left(\frac{q_t}{q_\infty}\right)^3 C_D$$

C_D, the turbine load factor is defined as

$$C_D=\frac{P_1-P_2}{\frac{1}{2}\rho q_t{}^2}$$

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where P, ρ, q, and A are pressure, density, velocity, and area, respectively. Indexes 1, 2, t, ∞ indicate conditions at the turbine entrance, the turbine exit, at the turbine, and the freestream conditions, respectively. It is apparent from Fig. 1 that the power obtained from the shrouded turbine can be as high as 3.5 times the maximum power available from an ideal turbine, of equal diameter, when both operate under the same freestream conditions.

It should be noted that for the considered shroud-and-turbine configuration, there is an optimal value of C_D. Furthermore, the maximum turbine power (maximum r) is achieved when the shroud is set at a yaw angle.

However, aside from its aerodynamic advantages, this shroud configuration is unacceptable for commercial use due to its high ratio of overall length to turbine diameter (about 7:1). As a result, various ways to shorten the shroud length without sacrificing performance were considered.

Present Work

Since the shroud is nothing more than a circular wing (high shroud performance is equivalent to high lift), shorter shrouds having the necessary performance can be designed by employing criteria similar to those used for high-lift producing wings. For example, the shrouds can have the longitudinal crosssection of high-lift airfoils, rather than the shape used in the past (see Fig. 1), and one can add axisymmetric flaps.[8] In the following, a brief summary of results obtained for shrouds designed according to these criteria is given. (The geometry of some of the tested models is shown in Fig. 2; further details are available in Ref. 8.)

Past work has indicated that the location of the axisymmetric flaps relative to the shroud exit plane has significant effect on the results obtained. It was shown that optimum performance is obtained when the radial gap between the circular flap and the shroud is approximately 4% of the shroud length and when there is no axial gap between the two.[7b,8]

Unlike the results of Refs. 5-7, the present experiments were conducted in the new wind tunnel of the Israel Aircraft Industry, with a test crosssection of 2.6 m × 3.6 m. Obviously, wall interference and blockage effects can be completely ignored when using the models of the size shown in Fig. 2 inside this tunnel. Therefore, based on the results shown in Fig. 3, we can say with complete confidence that a shrouded aerogenerator will perform much better than a bare propeller, even when the latter is aligned so as to point into the wind direction, while the former is rigidly fixed, provided the wind direction does not change more than within ±25° measured relative to the shroud axis and both the shrouded and the bare propeller have the same diameter. However, when θ increases beyond the value appropriate to r_{max}, a rapid reduction in r follows. As was mentioned before, this peculiar response of the shroud to yaw is not surprising since the shrouds considered are precisely circular wings. Consequently, the high values of r are directly associated with the lift of the shroud crosssection (airfoil). Earlier work by Fletcher[9] supports these findings. Fletcher, in his experimental work with circular wings, has shown that such airfoils (which he calls annular airfoils) have a large lift-drag ratio as compared to a planar wing and the the lift-curve slope of the circular wing is approximately twice the slope of a rectangular wing with the same aspect ratio. Therefore, it can be expected that changing the yaw angle for the shroud will increase the lift produced (and therefore the power ratio, r) until the stall angle is reached. The stall angle itself will be somewhat larger than the appropriate one for a similar rectangular wing.

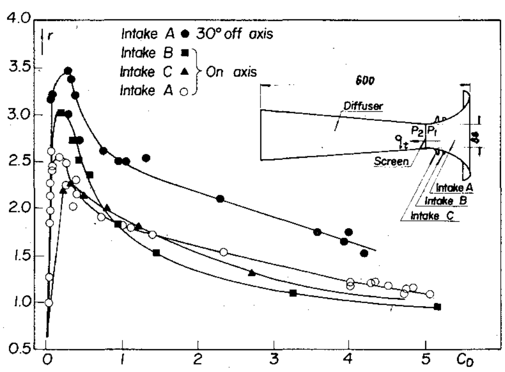

Fig. 1 Variation of r with C_D, all intakes.[6]

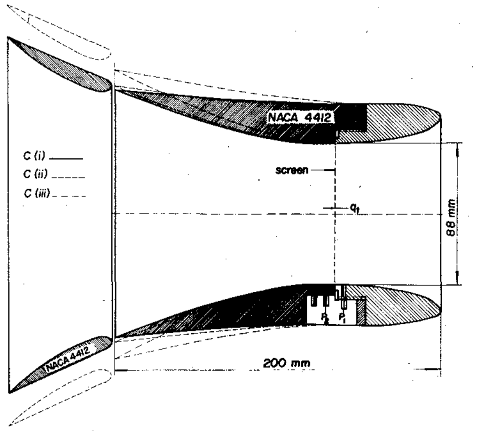

Fig. 2 Models C(i), C(ii), and C(iii).

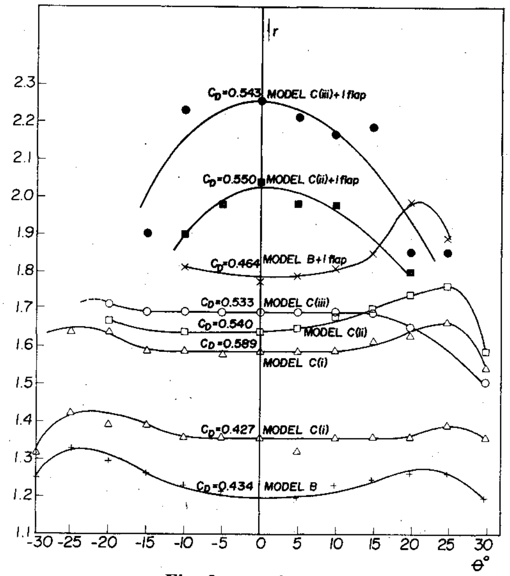

Fig. 3 r vs θ.

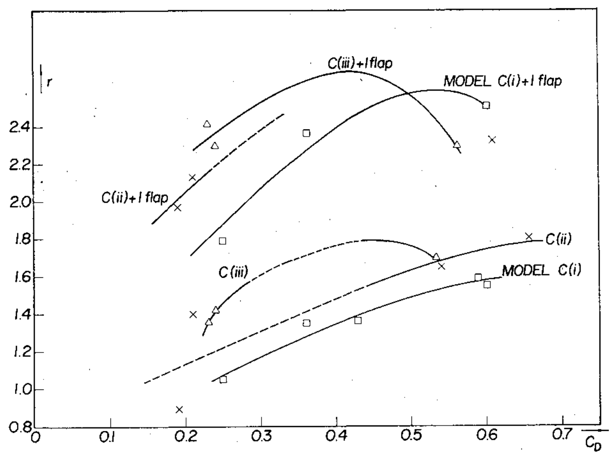

Fig. 4 r vs C_D at a zero yaw angle.

Figure 4 summarizes the obtained results for r in the $r - C_D$ plane. It is apparent that for a wide range of C_D's, an increase in the shroud exit area A_e results in an increase in r. Adding a circular flap to the shroud further increases the shroud's exit area A_e and, as expected, also increases r. Every shroud has its own optimal load factor C_D; in general, an increase in the shroud exit area (with or without flaps) reduces the optimal value of C_D (see Fig. 4).

Conclusions

The main object of this work was to arrive at a compact shroud configuration. The shrouds of models, C (i), C (ii), and C (iii), had "economical geometry," i.e., a fairly short length relative to the turbine diameter (total length to throat diameter ratio of the order of 3:1). Furthermore, when the wind direction changes, an increase in r can be expected in the shrouded aerogenerator as long as the yaw angles are within the shroud's stall range (about $\pm 25°$). Increasing the shroud exit area will increase the power ratio obtained under given stream conditions. The addition of a circular wing (flap) causes a significant improvement in the shroud performance (increase in r) but reduces significantly the stall range of the shroud. (Now r_{max} will be reached at $\theta = 0°$, see Fig. 3.) Three additional advantages of the use of shrouds for aerogenerators are 1) the axial velocity at the turbine section is higher than the freestream velocity, thereby making it possible to build smaller rotors that rotate at higher rpm; 2) by virtue of the enclosure produced by the shroud, tip losses can be significantly reduced; and 3) axial flow turbines are appropriate for use with shrouds. It has been shown that such turbines are capable of producing stable output at varying wind velocities without requiring a pitch control mechanism.[10]

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Generalized Inverse of a Matrix: The Minimization Approach

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Introduction

THE generalized inverse of a matrix[1] has been used extensively in the areas of modern control, least-square estimation, and aircraft structural analysis. In a recent paper,[2] certain practical aspects of the generalized inverse were discussed. It is the purpose of this engineering note to extend the results of Ref. 2 by presenting a unified framework that provides geometric insight and highlights certain optimal features imbedded in the generalized inverse.

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Consider the algebraic matrix equation

$$y = Ax \tag{1}$$

where A is an $n \times m$ constant matrix, y is a given n vector, and x is an m vector to be determined. For the trivial case where $n = m$ and A is a nonsingular matrix, i.e., rank $(A) = n$, a unique solution to Eq. (1) exists and is given by

$$x = A^{-1} y \tag{2}$$

where A^{-1} designates the inverse of A.

For the case $n \neq m$, the expression of x in terms of y involves the generalized inverse of A, denoted A^+, and thus

$$x = A^+ y \tag{3}$$

In the following cases it will be shown that A^+, for either $n > m$ or $n < m$, may be viewed as a solution to a certain minimization problem.

Case A: $n > m$

With no loss of generality it can be assumed that A is of full rank, i.e.,

$$\text{rank}(A) = m \tag{4}$$

If, however, rank $(A) < m$, it is possible to delete the dependent columns of A, set the respective unknown components of x equal to zero, and reduce the problem to the case in which Eq. (4) is satisfied. Since the m columns of A do not span the n dimensional space R_n, an exact solution to Eq. (1) cannot be obtained if y is not contained in the subspace spanned by the columns of A. Thus, one is motivated to seek approximate solutions, the best of which is the one that minimizes the Euclidian norm of the error. Let the error e be given by

$$e = y - Ax \tag{5}$$

Then let z be given by

$$z = \|e\|^2 = e^T e = (y - Ax)^T (y - Ax) \tag{6}$$

where the superscript T denotes the transpose. Evaluation of the gradient of z with respect to x yields

$$\partial z / \partial x = -2A^T y + 2A^T A x = 0 \tag{7}$$

while the Hessian matrix is

$$\partial^2 z / \partial x^2 = 2A^T A \tag{8}$$

From Eq. (7), x is given by

$$x = (A^T A)^{-1} A^T y \tag{9}$$

By virtue of A having full rank, $(A^T A)$ is a positive definite matrix. Thus $(A^T A)^{-1}$ exists and the Hessian matrix is positive definite, implying that a minimum was obtained. In this case $(n > m)$ the generalized inverse of A is given by

$$A^+ = (A^T A)^{-1} A^T \tag{10}$$

It is interesting to note that if y is contained in the subspace spanned by the columns of A, Eq. (9) yields an exact solution to Eq. (1), i.e., $\|e\| = 0$. The optimal feature of Eq. (9) has